Off-diagonal long-range order in a supersymmetric integrable model of correlated electrons*

J. Links[1,a], K. Hibberd[1], M. Gould[1], and A. Foerster[2]

[1] Department of Mathematics, University of Queensland, St Lucia, QLD, 4072, Australia
[2] Instítuto de Física da UFRGS, Av Bento Gonçalves 9500, Porto Alegre, RS, Brazil

Abstract. A new model for correlated electrons is presented which is integrable in one-dimension. The symmetry algebra of the model is the Lie superalgebra $gl(2|1)$ which depends on a continuous free parameter. This symmetry algebra contains the η pairing algebra as a subalgebra which is used to show that the model exhibits Off-Diagonal Long-Range Order in any number of dimensions.

PACS. 75.10.Jm Quantized spin models – 71.27.+a Strongly correlated electron systems; heavy fermions

The study of correlated electron models has attracted great interest largely motivated by the aim to understand high-T_c superconductivity. In a paper of Bariev, Klümper and Zittartz [1], a new model of correlated electrons was given which generalizes the Hubbard model with the introduction of correlated hopping and pair hopping terms. This model also generalized the supersymmetric U model given in [2] through the introduction of a spin anisotropy parameter.

Electrons on a lattice are described by canonical Fermi operators $c_{i,\sigma}$ and $c^\dagger_{i,\sigma}$ satisfying the anti-commutation relations given by

$$\{c^\dagger_{i,\sigma}, c_{j,\tau}\} = \delta_{ij}\delta_{\sigma\tau},$$

where $i, j = 1, 2, \cdots, L$ label the sites of the lattice and $\sigma, \tau = \uparrow, \downarrow$. The operator $c_{i,\sigma}$ annihilates an electron of spin σ at site i, which implies that the Fock vacuum $|0\rangle$ satisfies $c_{i,\sigma}|0\rangle = 0$. At a given lattice site i there are four possible electronic states:

$$|0\rangle, \ |\uparrow\rangle_i = c^\dagger_{i,\uparrow}|0\rangle, \ |\downarrow\rangle_i = c^\dagger_{i,\downarrow}|0\rangle, \ |\uparrow\downarrow\rangle_i = c^\dagger_{i,\downarrow}c^\dagger_{i,\uparrow}|0\rangle. \tag{1}$$

By $n_{i,\sigma} = c^\dagger_{i,\sigma}c_{i,\sigma}$ we denote the number operator for electrons with spin σ on site i, and we write $n_i = n_{i,\uparrow} + n_{i,\downarrow}$. The local spin operators are as follows:

$$S^\dagger_i = c^\dagger_{i,\uparrow}c_{i,\downarrow}, \ S_i = c^\dagger_{i,\downarrow}c_{i,\uparrow}, \ S^z_i = 1/2(n_{i,\uparrow} - n_{i,\downarrow}). \tag{2}$$

The Hamiltonian of the model as given in [1] reads

$$H = -\sum_{j,\sigma}(c^\dagger_{j\sigma}c_{j+1\sigma} + h.c.)\exp\left[-\frac{1}{2}(\eta - \sigma\gamma)n_{j,-\sigma}\right.$$
$$\left. -\frac{1}{2}(\eta + \sigma\gamma)n_{j+1,-\sigma}\right]$$
$$+ \sum_j\left[Un_{j\uparrow}n_{j\downarrow} + t(c^\dagger_{j\uparrow}c^\dagger_{j\downarrow}c_{j+1\downarrow}c_{j+1\uparrow} + h.c.)\right], \tag{3}$$

where j denotes the sites. This model with periodic boundary conditions was also solved by the co-ordinate Bethe ansatz under the constraint

$$t = \frac{U}{2} = \epsilon\left[2e^{-\eta}(\cosh\eta - \cosh\gamma)\right]^{\frac{1}{2}}, \quad \epsilon = \pm 1, \tag{4}$$

leaving two free parameters η, γ. It was subsequently shown [3] that this model, with the addition of an appropriate chemical potential term, could be derived from a $U_q(gl(2|1))$ invariant R-matrix which satisfies the Yang-Baxter equation, thus establishing integrability of the model by virtue of the Quantum Inverse Scattering Method (QISM) (for example, see [4]). The free parameters η, γ are functions of the deformation parameter q and the continuous parameter labelling the inequivalent four-dimensional typical representations of $U_q(gl(2|1))$.

An important concept in the theory of high-T_c superconductivity is that of Off-Diagonal Long-Range Order (ODLRO), orginally introduced by Yang [5] who showed that it was a necessary feature of systems which exhibit superconducting states. An elegant method of constructing states with ODLRO is through the use of the η pairing realization of the Lie algebra $sl(2)$. Already this technique has been applied to a variety of models [6–10]. Our intention here is to propose a new model of correlated electrons

---

from the aforementioned R-matrix with the η pairing symmetry and to show that states with ODLRO do exist. We work in the isotropic case ($q = 1$), otherwise we would necessarily need to adopt a q deformation of the η pairing algebra. Also it is known that quantum superalgebra invariance is broken by the imposition of periodic boundary conditions but is restored as $q \to 1$. Here we choose a new realization of the abstract model used to verify integrability of the Hamiltonian (3). The spectra of this new model and the isotropic limit of (3) are identical since both are derived from the same solution of the Yang-Baxter equation. However the physics of these two models is different in terms of their correlation functions since the creation and annihilation operators for electrons are represented by different matrices in each case. Importantly, for the present model a simple study of the off-diagonal elements of the reduced density matrix indicates ODLRO.

The local Hamiltonian for the new model is derived below and has the following form:

$$
h_{ij}(\alpha) = \left[-(\alpha+1)(c_{i,\uparrow}^\dagger c_{j,\uparrow} + c_{j,\uparrow}^\dagger c_{i,\uparrow}) \left(\frac{\alpha}{\alpha+1} \right)^{\frac{1}{2}(n_{i,\downarrow}+n_{j,\downarrow})} \right.
$$
$$
+ \alpha(c_{i,\downarrow}^\dagger c_{j,\downarrow} + c_{j,\downarrow}^\dagger c_{i,\downarrow}) \left(\frac{\alpha+1}{\alpha} \right)^{\frac{1}{2}(n_{i,\uparrow}+n_{j,\uparrow})}
$$
$$
- (n_{i,\uparrow}n_{i,\downarrow} + n_{j,\uparrow}n_{j,\downarrow}) - \alpha(n_{i,\downarrow} + n_{j,\downarrow})
$$
$$
\left. + (\alpha+1)(n_{i,\uparrow} + n_{j,\uparrow}) + S_i^\dagger S_j + S_i S_j^\dagger \right] \qquad (5)
$$

where i, j denote nearest neighour sites on the lattice. Above we adopt the convention $0 \le \arg z^{1/2} < \pi$ for any complex parameter z. Here α is a free parameter which we will restrict to being real.

The Hamiltonian describes correlated hopping processes, a Hubbard on-site interaction, chemical potential and an XY spin interaction. The energies are given by

$$
E = \alpha \sum_{j=1}^{N} \frac{1}{\mu_j^2 + 1/4} - 2\alpha L
$$

corresponding to a solution of the Bethe ansatz equations [1,11–14]

$$
\left[\frac{\mu_j - \frac{i}{2}}{\mu_j + \frac{i}{2}} \right]^L = \prod_{k=1}^{M} \frac{\mu_j - \lambda_k - \frac{i}{2(\alpha+1)}}{\mu_j - \lambda_k + \frac{i}{2(\alpha+1)}},
$$
$$
j = 1...N,
$$
$$
\prod_{j=1}^{N} \frac{\lambda_k - \mu_j - \frac{i}{2(\alpha+1)}}{\lambda_k - \mu_j + \frac{i}{2(\alpha+1)}} = -\prod_{\beta=1}^{M} \frac{\lambda_k - \lambda_\beta - \frac{i}{(\alpha+1)}}{\lambda_k - \lambda_\beta + \frac{i}{(\alpha+1)}},
$$
$$
k = 1, ..., M.
$$

In the above L is the length of the system, M is the number of spin up electrons and $N = L + 2S^z$. We mention that an alternative set of Bethe ansatz equations and expressions for the energies have recently been obtained in [15].

The eigenstates corresponding to solutions of the above Bethe ansatz do not provide a complete set of states for the model. However each gives the lowest weight state for a $gl(2|1)$ multiplet allowing additional states to be obtained through the action of the $gl(2|1)$ generators. The method used to show this is analogous to that used for the $t - J$ model in [16]. Note under spin reflection we have $h(\alpha) \to h(-\alpha - 1)$ so that α can be interpreted as a measure of the spin anisotropy coupling of the model. We do not include the isotropic case $\alpha = -1/2$ here since taking the limit is non-trivial. A Bethe ansatz solution of this case has been studied in [17].

We denote the generators of $gl(2|1)$ by E_γ^β, $\beta, \gamma = 1, 2, 3$ with grading $[1] = [2] = 0$, $[3] = 1$. In the typical 4-dimensional representation of $gl(2|1)$, the highest weight itself of the representation depends on a free parameter α, thus giving rise to a one-parameter family of inequivalent irreps [18]. Choose the following basis

$$
|4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},
$$
$$
|2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad (6)
$$

with $|1\rangle$, $|4\rangle$ even and $|2\rangle$, $|3\rangle$ odd. Then in this typical 4-dimensional representation, E_γ^β are 4×4 supermatrices of the form

$$
E_2^1 = |2\rangle \langle 3|,
$$
$$
E_1^2 = |3\rangle \langle 2|,
$$
$$
E_1^1 = -|3\rangle \langle 3| - |4\rangle \langle 4|,
$$
$$
E_2^2 = -|2\rangle \langle 2| - |4\rangle \langle 4|,
$$
$$
E_3^2 = \sqrt{\alpha} |1\rangle \langle 2| + \sqrt{\alpha+1} |3\rangle \langle 4|,
$$
$$
E_2^3 = \sqrt{\alpha} |2\rangle \langle 1| + \sqrt{\alpha+1} |4\rangle \langle 3|,
$$
$$
E_3^1 = -\sqrt{\alpha} |1\rangle \langle 3| + \sqrt{\alpha+1} |2\rangle \langle 4|,
$$
$$
E_1^3 = -\sqrt{\alpha} |3\rangle \langle 1| + \sqrt{\alpha+1} |4\rangle \langle 2|,
$$
$$
E_3^3 = \alpha |1\rangle \langle 1| + (\alpha+1) (|2\rangle \langle 2| + |3\rangle \langle 3|)
$$
$$
+ (\alpha+2) |4\rangle \langle 4|. \qquad (7)
$$

For the case $0 > \alpha > -1$, the above representation equation (7) is grade-star [19]. Consequently the Hamiltonian equation (5) is not Hermitian but rather exhibits grade star Hermiticity. For other real values of α the Hamiltonian is Hermitian in the usual sense. For the tensor product decomposition we have ($\alpha \ne 0, -1/2, -1$) $V \otimes V = V_1 \oplus V_2 \oplus V_3$, where V_1, V_2 and V_3 are $gl(2|1)$-modules with highest weights $(0, 0|2\alpha)$, $(0, -1|2\alpha+1)$, and $(-1, -1|2\alpha+2)$.

The rational $\check{R}(\theta) \in \text{End}(V \otimes V)$ matrix satisfying the supersymmetric Yang-Baxter equation

$$
[I \otimes \check{R}(\theta)][\check{R}(\theta + \theta') \otimes I][I \otimes \check{R}(\theta')] =
$$
$$
[\check{R}(\theta') \otimes I][I \otimes \check{R}(\theta + \theta')][\check{R}(\theta) \otimes I], \quad (8)
$$

is given by [20]

$$\check{R}(\theta) = -\frac{\theta - 2\alpha}{\theta + 2\alpha}P_1 + P_2 - \frac{\theta + 2\alpha + 2}{\theta - 2\alpha - 2}P_3 \qquad (9)$$

where P_k, $k = 1, 2, 3$, are projection operators from $V \otimes V$ onto V_k. Throughout the multiplication rule for the tensor product is defined by

$$(a \otimes b)(c \otimes d) = (-1)^{[b][c]} (ac \otimes bd) \qquad (10)$$

for homogeneous elements a, b, c and d.

The projectors P_k can easily be evaluated:

$$\begin{aligned}
P_1 &= \left|\Psi_1^1\right\rangle\left\langle\Psi_1^1\right| + \left|\Psi_2^1\right\rangle\left\langle\Psi_2^1\right| \\
&\quad + \left|\Psi_3^1\right\rangle\left\langle\Psi_3^1\right| + \left|\Psi_4^1\right\rangle\left\langle\Psi_4^1\right|, \\
P_3 &= \left|\Psi_1^3\right\rangle\left\langle\Psi_1^3\right| + \left|\Psi_2^3\right\rangle\left\langle\Psi_2^3\right| \\
&\quad + \left|\Psi_3^3\right\rangle\left\langle\Psi_3^3\right| + \left|\Psi_4^3\right\rangle\left\langle\Psi_4^3\right|, \\
P_2 &= I - P_1 - P_3
\end{aligned} \qquad (11)$$

where $\left|\Psi_k^1\right\rangle$ and $\left|\Psi_k^3\right\rangle$, $k = 1, 2, 3, 4$, form the symmetry adapted bases for the spaces V_1 and V_3, respectively. Note that $\check{R}(0) \equiv I$. We now compute $\left|\Psi_k^1\right\rangle$ and $\left|\Psi_k^3\right\rangle$, $k = 1, 2, 3, 4$. By means of the matrix representation (equation (7)), one can show

$$\begin{aligned}
\left|\Psi_1^1\right\rangle &= |1\rangle \otimes |1\rangle, \\
\left|\Psi_2^1\right\rangle &= \frac{1}{\sqrt{2}}(|2\rangle \otimes |1\rangle + |1\rangle \otimes |2\rangle), \\
\left|\Psi_3^1\right\rangle &= \frac{1}{\sqrt{2}}(|3\rangle \otimes |1\rangle + |1\rangle \otimes |3\rangle), \\
\left|\Psi_4^1\right\rangle &= \frac{1}{\sqrt{2(2\alpha+1)}}[\sqrt{\alpha+1}(|4\rangle \otimes |1\rangle + |1\rangle \otimes |4\rangle) \\
&\quad + \sqrt{\alpha}(|2\rangle \otimes |3\rangle - |3\rangle \otimes |2\rangle)], \\
\left|\Psi_1^3\right\rangle &= \frac{1}{\sqrt{2(2\alpha+1)}}[\sqrt{\alpha}(|4\rangle \otimes |1\rangle + |1\rangle \otimes |4\rangle) \\
&\quad + \sqrt{\alpha+1}(-|2\rangle \otimes |3\rangle + |3\rangle \otimes |2\rangle)], \\
\left|\Psi_2^3\right\rangle &= \frac{1}{\sqrt{2}}(|2\rangle \otimes |4\rangle + |4\rangle \otimes |2\rangle), \\
\left|\Psi_3^3\right\rangle &= \frac{1}{\sqrt{2}}(|3\rangle \otimes |4\rangle + |4\rangle \otimes |3\rangle), \\
\left|\Psi_4^3\right\rangle &= |4\rangle \otimes |4\rangle
\end{aligned} \qquad (12)$$

with the dual basis elements defined by

$$\begin{aligned}
&\left\langle\Psi_k^1\right| = \left(\left|\Psi_k^1\right\rangle\right)^\dagger, \quad \left\langle\Psi_k^3\right| = \left(\left|\Psi_k^3\right\rangle\right)^\dagger, \quad k = 1, 2, 3, 4, \\
&(|\beta\rangle \otimes |\gamma\rangle)^\dagger = (|\beta\rangle)^\dagger \otimes (|\gamma\rangle)^\dagger, \quad 0 > \alpha > -1 \\
&(|\beta\rangle \otimes |\gamma\rangle)^\dagger = (-1)^{[\beta][\gamma]}(|\beta\rangle)^\dagger \otimes (|\gamma\rangle)^\dagger, \\
&\alpha > 0, \ \alpha < -1 \\
&(|\beta\rangle)^\dagger = \langle\beta|, \qquad \forall \beta = 1, 2, 3, 4.
\end{aligned} \qquad (13)$$

Here $[\beta]$ stands for the grading of the state $|\beta\rangle$: $[\beta] = 0$ for even $|\beta\rangle$ and $[\beta] = 1$ for odd $|\beta\rangle$.

We may define the local Hamiltonian [4]

$$h(\alpha) = -2\alpha(\alpha+1)\frac{d}{d\theta}\check{R}(\theta)\Big|_{\theta=0} = 2(\alpha+1)P_1 - 2\alpha P_3. \quad (14)$$

We remark that for $\alpha = -1/2$ the form equation (14) is no longer valid [17]. A realization of this local Hamiltonian by choosing

$$|1\rangle \equiv |\uparrow\rangle, \quad |2\rangle \equiv |\uparrow\downarrow\rangle, \quad |3\rangle \equiv |0\rangle, \quad |4\rangle \equiv |\downarrow\rangle \quad (15)$$

yields the local Hamiltonian equation (5). Choosing the alternative realization

$$|1\rangle \equiv |\uparrow\downarrow\rangle, \quad |2\rangle \equiv |\downarrow\rangle, \quad |3\rangle \equiv |\uparrow\rangle, \quad |4\rangle \equiv |0\rangle \quad (16)$$

yields the isotropic limit of (3) with

$$\exp\gamma = 1, \quad \exp(-\eta) = \frac{\alpha+1}{\alpha}$$

and the chemical potential term

$$2\sum_j(n_{j\uparrow} + n_{j\downarrow}).$$

The new Hamiltonian (5) is thus obtained from (3) by the unitary transformation

$$\begin{aligned}
c_\uparrow &\to c_\downarrow^\dagger(1 - 2n_\uparrow) \\
c_\downarrow &\to c_\uparrow(1 - 2n_\downarrow)
\end{aligned}$$

It is easy to verify that $H|0\rangle = 0$ where $|0\rangle$ denotes the vacuum state. By construction, we know that the Hamiltonian has $gl(2|1)$ invariance and moreover the $sl(2)$ sub-algebra is represented by the η pair realization; viz

$$\eta = \sum_{j=1}^{L} c_{j,\uparrow}c_{j,\downarrow}, \quad \eta^\dagger = \sum_{j=1}^{L} c_{j,\downarrow}^\dagger c_{j,\uparrow}^\dagger,$$

$$\eta^z = \sum_{j=1}^{L} \frac{1}{2}(n_j - 1). \qquad (17)$$

Thus the $2N$ electron states

$$|\Psi_N\rangle = (\eta^\dagger)^N|0\rangle \qquad (18)$$

are eigenstates of the global Hamiltonian with zero energy. These states are well known to possess ODLRO; that is

$$\lim_{|l-j|\to\infty} \frac{\langle\Psi_N| c_{j,\downarrow}^\dagger c_{j,\uparrow}^\dagger c_{l,\uparrow}c_{l,\downarrow} |\Psi_N\rangle}{\langle\Psi_N|\Psi_N\rangle} = \frac{N}{L}\left(1 - \frac{N}{L}\right) \quad (19)$$

in the thermodynamic limit $(N, L \to \infty, \ N/L$ fixed$)$ in any number of dimensions [10]. In one dimension, these eigenstates belong to the multiplet generated by the $gl(2|1)$ lowest weight state corresponding to the solution of the Bethe ansatz equations with $N = L - 1$, $M = 0$ and

$$\mu_k = \frac{1}{2}\cot(\frac{\pi k}{L}), \quad k = 1, ..., L - 1.$$

The Hermitian regions $\alpha > 0$ and $\alpha < -1$ have ground state energies $-2L\alpha$ and $2L(\alpha+1)$ respectively which follows from the action of the Hamiltonian on the completely ferromagnetic reference states and the fact that

$$E \geq LE_0$$

where E_0 is the minimum energy of the two-site Hamiltonian. Here it is clear that the $2N$ electron states (18) do not belong to the ground state multiplet. On the other hand for $0 > \alpha > -1$, which is the region of small spin anisotropic coupling, the states (18) do occur in the ground state at least for small lattices and generic values of α. Unfortunately technical difficulties arise in extending this result to the thermodynamic limit due to the grade Hermitian nature of the Hamiltonian. New mathematical methods need to be developed in order to understand the behaviour for these values of α in the infinite limit, which are currently under investigation.

References

1. R. Bariev, A. Klümper, J. Zittartz, Europhys. Lett. **32**, 85 (1995).
2. A.J. Bracken, M.D. Gould, J.R. Links, Y.-Z. Zhang, Phys. Rev. Lett. **74**, 2768 (1995).
3. M.D. Gould, K.E. Hibberd, J.R. Links, Y.-Z. Zhang, Phys. Lett. A **212**, 156 (1996).
4. P.P. Kulish, E.K. Sklyanin, J. Sov. Math. **19**, 1596 (1982).
5. C.N.Yang, Rev. Mod. Phys. **34**, 694 (1962).
6. C.N. Yang, Phys. Rev. Lett. **63**, 2144 (1989).
7. F.H.L. Essler, V.E. Korepin, K. Schoutens, Phys. Rev. Lett. **68**, 2960 (1992); **70**, 73 (1993).
8. L. Arrachea, A.A. Aligia, Phys. Rev. Lett. **73**, 2240 (1994).
9. A. Schadschneider, Phys. Rev. B **51**, 10386 (1995).
10. J. de Boer, V.E. Korepin, A. Schadschneider, Phys. Rev. Lett. **74**, 789 (1995).
11. Z. Maassarani, J. Phys. A **28**, 1305 (1995).
12. G. Bedürftig, H. Frahm, J. Phys. A **28**, 4453 (1995).
13. K.E. Hibberd, M.D. Gould, J.R. Links, Phys. Rev. B **54**, 8430 (1996).
14. P.B. Ramos, M.J. Martins, Nucl. Phys. B **474**, 678 (1996).
15. M.P. Pfannmuller, H. Frahm, J. Phys. A **30**, L543 (1997).
16. A. Foerster, M. Karowski, Nucl. Phys. B **396**, 611 (1993).
17. M.J. Martins, P.B. Ramos, J. Phys. A **28**, L525 (1995).
18. V.G. Kac, Adv. Math. **26**, 8 (1977).
19. M.D. Gould, R.B. Zhang, J. Math. Phys. **31**, 1524 (1990).
20. A.J. Bracken, G.W. Delius, M.D. Gould, Y.-Z. Zhang, J. Phys. A **27**, 6551 (1994).